# Equivalence of CFGs and PDAs

## Lecture 22
## Section 7.2

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Fri, Oct 14, 2016

# Outline


1. Equivalence of PDAs and CFGs
   - Proof $\Leftarrow$
2. An Example
   - Generate the Grammar
   - Simplify the Grammar
3. A Second Example
4. Assignment

# Outline


1. Equivalence of PDAs and CFGs
   - Proof ⇐
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# Equivalence of PDAs and CFGs

**Theorem (Equivalence of PDAs and CFGs)**

- *If $G$ is a CFG, then there exists a PDA $M$ such that $L(G) = L(M)$.*
- *If $M$ is a PDA, then there exists a CFG $G$ such that $L(M) = L(G)$.*

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# Equivalence of PDAs and CFGs

**Proof (⇐).**

- Given a PDA $M$, we must construct a grammar $G$ that generates $L(M)$.
- Modify $M$ so that
  - $M$ has a single accept state.
  - $M$ empties its stack before accepting.
  - Each transition either pushes one symbol or pops one symbol, but not both.

□

# The Variables

**Proof ($\Leftarrow$).**

- Every transition is of the form

$$\delta(p, a, A) = (q, \lambda)$$

or

$$\delta(p, a, A) = (q, BC)$$

where $p, q \in Q$, $a \in \Sigma \cup \{\lambda\}$, and $A, B, C \in \Gamma$.

□

# The Variables

**Proof ($\Leftarrow$).**

- For $p, q \in Q$ and for all $A \in \Gamma$, we create a variable

$$(pAq).$$

- The variable $(pAq)$ is interpreted to mean

  "We can get from state $p$ to state $q$ and, in the process, the net effect is to remove $A$ from the stack."

- If it is obviously impossible to get from $p$ to $q$ at all, then we may disregard all variables $(pAq)$.

$\square$

# The Variables

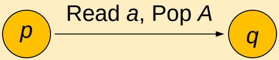
**Proof ($\Leftarrow$).**

- Transitions of the form

$$\delta(p, a, A) = (q, \lambda)$$

  will produce productions of the form

$$(pAq) \to a.$$




□

# The Variables

## Proof ($\Leftarrow$).

- Transitions of the form

$$\delta(p, a, A) = (q, BC)$$

will produce productions of the form

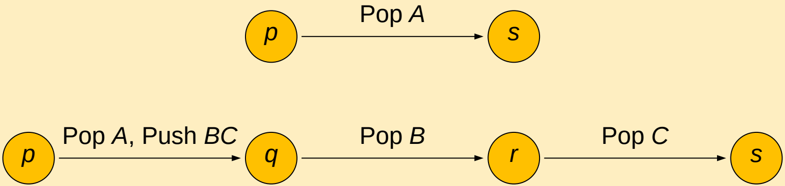
$$(pAs) \to a(qBr)(rCs)$$

for all possible choices of states $r$ and $s$.

$\square$

# The Variables

**Proof (⇐).**




□

# The Grammar Rules

**Proof (⇐).**

- Since the stack starts with $\mathbf{z}$ and ends empty, our start symbol is $(q_0\mathbf{z}q_f)$.

□

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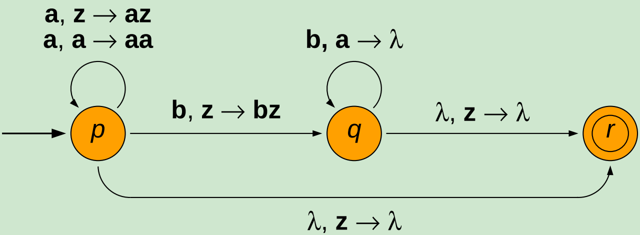
# Example

## Example (Convert a PDA to a CFG)

- Find a grammar for the following PDA.



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# Example

## Example (Convert a PDA to a CFG)

- $Q = \}p, q, r\}$ and $\Gamma = \{\mathbf{a}, \mathbf{z}\}$, so the variables are

$$\begin{array}{ccc}
(p\mathbf{a}p) & (q\mathbf{a}p) & (r\mathbf{a}p) \\
(p\mathbf{a}q) & (q\mathbf{a}q) & (r\mathbf{a}q) \\
(p\mathbf{a}r) & (q\mathbf{a}r) & (r\mathbf{a}r) \\
\hline
(p\mathbf{z}p) & (q\mathbf{z}p) & (r\mathbf{z}p) \\
(p\mathbf{z}q) & (q\mathbf{z}q) & (r\mathbf{z}q) \\
(p\mathbf{z}r) & (q\mathbf{z}r) & (r\mathbf{z}r)
\end{array}$$

# Example

## Example (Convert a PDA to a CFG)

- However, in this example, it is impossible to go from $q$ to $p$, from $r$ to $p$, or from $r$ to $q$.
- So we may eliminate $(qxp)$, $(rxp)$, and $(rxq)$, for all $x \in \Gamma$.
- The remaining variables are

$$\begin{array}{ccc} (p\mathbf{a}p) & & \\ (p\mathbf{a}q) & (q\mathbf{a}q) & \\ (p\mathbf{a}r) & (q\mathbf{a}r) & (r\mathbf{a}r) \\ \hline (p\mathbf{z}p) & & \\ (p\mathbf{z}q) & (q\mathbf{z}q) & \\ (p\mathbf{z}r) & (q\mathbf{z}r) & (r\mathbf{z}r) \end{array}$$

# The Grammar Rules (Group 1)

## Example (Convert a PDA to a CFG)

- The transitions of the first kind are

$$\delta(p, \lambda, \mathbf{z}) = (r, \lambda)$$
$$\delta(p, \mathbf{b}, \mathbf{a}) = (q, \lambda)$$
$$\delta(q, \mathbf{b}, \mathbf{a}) = (q, \lambda)$$
$$\delta(q, \lambda, \mathbf{z}) = (r, \lambda)$$

# The Grammar Rules (Group 1)

## Example (Convert a PDA to a CFG)

- These give us the productions

$$(p\mathbf{z}r) \to \lambda$$
$$(p\mathbf{a}q) \to \mathbf{b}$$
$$(q\mathbf{a}q) \to \mathbf{b}$$
$$(q\mathbf{z}r) \to \lambda$$

# The Grammar Rules (Group 1)

## Example (Convert a PDA to a CFG)

- The transitions of the second kind are

$$\delta(p, \mathbf{a}, \mathbf{z}) = (p, \mathbf{az})$$
$$\delta(p, \mathbf{a}, \mathbf{a}) = (p, \mathbf{aa})$$

# The Grammar Rules (Group 1)

## Example (Convert a PDA to a CFG)

- These give us the productions

$$(p\mathbf{z}s) \to \mathbf{a}(p\mathbf{a}t)(t\mathbf{z}s)$$
$$(p\mathbf{a}s) \to \mathbf{a}(p\mathbf{a}t)(t\mathbf{a}s)$$

for all possible choices of $s, t \in Q$.

# The Grammar Rules

## Example (Convert a PDA to a CFG)

- The production $(pzs) \to \mathbf{a}(pat)(tzs)$ represents

$$(pzp) \to \mathbf{a}(pap)(pzp)$$
$$(pzp) \to \mathbf{a}(paq)(qzp)$$
$$(pzp) \to \mathbf{a}(par)(rzp)$$
$$(pzq) \to \mathbf{a}(pap)(pzq)$$
$$(pzq) \to \mathbf{a}(paq)(qzq)$$
$$(pzq) \to \mathbf{a}(par)(rzq)$$
$$(pzr) \to \mathbf{a}(pap)(pzr)$$
$$(pzr) \to \mathbf{a}(paq)(qzr)$$
$$(pzr) \to \mathbf{a}(par)(rzr)$$

# The Grammar Rules

## Example (Convert a PDA to a CFG)

- But we eliminate the impossible ones, leaving

$$x(p\mathbf{z}p) \to \mathbf{a}(p\mathbf{a}p)(p\mathbf{z}p)$$
$$x(p\mathbf{z}q) \to \mathbf{a}(p\mathbf{a}p)(p\mathbf{z}q)$$
$$x(p\mathbf{z}q) \to \mathbf{a}(p\mathbf{a}q)(q\mathbf{z}q)$$
$$x(p\mathbf{z}r) \to \mathbf{a}(p\mathbf{a}p)(p\mathbf{z}r)$$
$$x(p\mathbf{z}r) \to \mathbf{a}(p\mathbf{a}q)(q\mathbf{z}r)$$
$$x(p\mathbf{z}r) \to \mathbf{a}(p\mathbf{a}r)(r\mathbf{z}r)$$

# The Grammar Rules

## Example (Convert a PDA to a CFG)

- Similarly, $(pas) \to \mathbf{a}(pat)(tas)$ produces

$$
\begin{aligned}
(pap) &\to \mathbf{a}(pap)(pap) \\
(paq) &\to \mathbf{a}(pap)(paq) \\
(paq) &\to \mathbf{a}(paq)(qaq) \\
(par) &\to \mathbf{a}(pap)(par) \\
(par) &\to \mathbf{a}(paq)(qar) \\
(par) &\to \mathbf{a}(par)(rar)
\end{aligned}
$$

# The Grammar Rules

## Example (Convert a PDA to a CFG)

- The grammar, with start symbol $(p\mathbf{z}r)$:

$$
\begin{aligned}
(p\mathbf{z}r) &\to \mathbf{a}(p\mathbf{a}p)(p\mathbf{z}r) \mid \mathbf{a}(p\mathbf{a}q)(q\mathbf{z}r) \mid \mathbf{a}(p\mathbf{a}r)(r\mathbf{z}r) \mid \lambda \\
(p\mathbf{z}p) &\to \mathbf{a}(p\mathbf{a}p)(p\mathbf{z}p) \\
(p\mathbf{z}q) &\to \mathbf{a}(p\mathbf{a}p)(p\mathbf{z}q) \mid \mathbf{a}(p\mathbf{a}q)(q\mathbf{z}q) \\
(p\mathbf{a}p) &\to \mathbf{a}(p\mathbf{a}p)(p\mathbf{a}p) \\
(p\mathbf{a}q) &\to \mathbf{a}(p\mathbf{a}p)(p\mathbf{a}q) \mid \mathbf{a}(p\mathbf{a}q)(q\mathbf{a}q) \mid \mathbf{b} \\
(p\mathbf{a}r) &\to \mathbf{a}(p\mathbf{a}p)(p\mathbf{a}r) \mid \mathbf{a}(p\mathbf{a}q)(q\mathbf{a}r) \mid \mathbf{a}(p\mathbf{a}r)(r\mathbf{a}r) \\
(q\mathbf{a}q) &\to \mathbf{b} \\
(q\mathbf{z}r) &\to \lambda
\end{aligned}
$$

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# The Grammar Rules

## Example (Convert a PDA to a CFG)

- Eliminate the production $(qzr) \to \lambda$:

$$
\begin{aligned}
(pzr) &\to \mathbf{a}(pap)(pzr) \mid \mathbf{a}(paq) \mid \mathbf{a}(par)(rzr) \mid \lambda \\
(pzp) &\to \mathbf{a}(pap)(pzp) \\
(pzq) &\to \mathbf{a}(pap)(pzq) \mid \mathbf{a}(paq)(qzq) \\
(pap) &\to \mathbf{a}(pap)(pap) \\
(paq) &\to \mathbf{a}(pap)(paq) \mid \mathbf{a}(paq)(qaq) \mid \mathbf{b} \\
(par) &\to \mathbf{a}(pap)(par) \mid \mathbf{a}(paq)(qar) \mid \mathbf{a}(par)(rar) \\
(qaq) &\to \mathbf{b}
\end{aligned}
$$

# The Grammar Rules

## Example (Convert a PDA to a CFG)

- Eliminate the production $(qaq) \to \mathbf{b}$:

$$
\begin{aligned}
(p\mathbf{z}r) &\to \mathbf{a}(p\mathbf{a}p)(p\mathbf{z}r) \mid \mathbf{a}(p\mathbf{a}q) \mid \mathbf{a}(p\mathbf{a}r)(r\mathbf{z}r) \mid \lambda \\
(p\mathbf{z}p) &\to \mathbf{a}(p\mathbf{a}p)(p\mathbf{z}p) \\
(p\mathbf{z}q) &\to \mathbf{a}(p\mathbf{a}p)(p\mathbf{z}q) \mid \mathbf{a}(p\mathbf{a}q)(q\mathbf{z}q) \\
(p\mathbf{a}p) &\to \mathbf{a}(p\mathbf{a}p)(p\mathbf{a}p) \\
(p\mathbf{a}q) &\to \mathbf{a}(p\mathbf{a}p)(p\mathbf{a}q) \mid \mathbf{a}(p\mathbf{a}q)\mathbf{b} \mid \mathbf{b} \\
(p\mathbf{a}r) &\to \mathbf{a}(p\mathbf{a}p)(p\mathbf{a}r) \mid \mathbf{a}(p\mathbf{a}q)(q\mathbf{a}r) \mid \mathbf{a}(p\mathbf{a}r)(r\mathbf{a}r) \\
(q\mathbf{a}q) &\to \mathbf{b}
\end{aligned}
$$

# The Grammar Rules

## Example (Convert a PDA to a CFG)

- Useless variables are $(p\mathbf{z}p)$ and $(p\mathbf{a}p)$.
- Eliminate them and all productions that use them.

$$
\begin{aligned}
(p\mathbf{z}r) &\to \mathbf{a}(p\mathbf{a}q) \mid \mathbf{a}(p\mathbf{a}r)(r\mathbf{z}r) \mid \lambda \\
(p\mathbf{z}q) &\to \mathbf{a}(p\mathbf{a}q)(q\mathbf{z}q) \\
(p\mathbf{a}q) &\to \mathbf{a}(p\mathbf{a}q)\mathbf{b} \mid \mathbf{b} \\
(p\mathbf{a}r) &\to \mathbf{a}(p\mathbf{a}q)(q\mathbf{a}r) \mid \mathbf{a}(p\mathbf{a}r)(r\mathbf{a}r) \\
(q\mathbf{a}q) &\to \mathbf{b}
\end{aligned}
$$

# The Grammar Rules

## Example (Convert a PDA to a CFG)

- The variables $(r\mathbf{z}r)$, $(q\mathbf{z}q)$, $(q\mathbf{a}r)$, and $(r\mathbf{a}r)$ never appear on the lefthand side of any production.
- Eliminate them and all productions that use them.

$$(p\mathbf{z}r) \to \mathbf{a}(p\mathbf{a}q) \mid \lambda$$
$$(p\mathbf{a}q) \to \mathbf{a}(p\mathbf{a}q)\mathbf{b} \mid \mathbf{b}$$
$$(q\mathbf{a}q) \to \mathbf{b}$$

# The Grammar Rules

## Example (Convert a PDA to a CFG)

- Now $(q\mathbf{a}q)$ is useless, so eliminate it.

$$(p\mathbf{z}r) \to \mathbf{a}(p\mathbf{a}q) \mid \lambda$$

$$(p\mathbf{a}q) \to \mathbf{a}(p\mathbf{a}q)\mathbf{b} \mid \mathbf{b}$$

# The Grammar Rules

## Example (Convert a PDA to a CFG)

- Finally, give the two remaining variables simple names $S$ and $A$.

$$S \to \mathbf{a}A \mid \lambda$$
$$A \to \mathbf{a}A\mathbf{b} \mid \mathbf{b}$$

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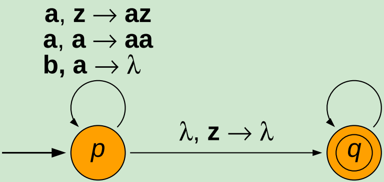
# A Second Example

## Example (Convert a PDA to a CFG)

- Find a grammar for the following PDA.



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## Assignment

- Section 7.2 Exercise 16.
- Simplify the grammar of Exercise 16.
- Find and simplify a grammar for the following PDA.